5. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1967.
6. Moiseev, N. N., Numerical Methods in the Theory of Optimal Systems. Moscow, "Nauka", 1971.
7. Chernous'ko, F. L., Certain problems of optimal control with a small parameter. PMM Vol. 32, №1, 1968.
8. Evtushenko, Iu. G., Approximate calculation of optimal control problems. PMM Vol. 34, №1, 1970.
9. Akulenko, L. D., Investigation of certain optimal systems by the averaging method. PMM Vol. 38, №3, 1974.
10. Gantmakher, F. R., Lectures on Analytic Mechanics. Moscow, "Nauka", 1966.
11. Samoilenko, A. M., Proof of the averaging principle for differential equations with a discontinuous right-hand side. In: Approximate Solution Methods for Differential Equations. Kiev, Izd. Akad. Nauk Ukrain. SSR, 1963.

Translated by N. H. C.

UDC 531.31

LIAPUNOV CANONIC TRANSFORMATIONS AND NORMAL HAMILTONIAN FORMS

PMM Vol. 39, №4, 1975, pp. 604-613
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(Received December 2, 1974)

We investigate to what general form can a Hamiltonian be reduced by an arbitrary canonic transformation preserving the property of Liapunov stability. We have succeeded in answering this question fully in the case of a stable autonomous Hamiltonian. One of the results of the analysis undertaken is a method of reducing the Hamiltonian to normal form in finite order, different from those proposed earlier [1], possessing definite advantages in comparison with them and exposing the connection between the methods of normalization and of averaging. We derive a table allowing us to compute from the original Hamiltonian its third-order normal form in the presence of any third-order resonances. A canonic transformation of the original Hamiltonian to a form more convenient for study is usually used in the investigation of the Liapunov stability of an equilibrium position. From such a viewpoint we can arrive at the method of Birkhoff transformations [2] and many stability results have recently been obtained in this way, having a practical value (for example, [3-5] and others). In the application of the method indicated it is necessary that there exist a close connection between the stability properties of the original and of the transformed Hamiltonians. Therefore, only autonomous transformations are usually used. However, such a restriction is not connected with the conditions for the applicability of the given method even in the case of an autonomous original Hamiltonian. It is interesting to consider this problem from a general point of view, without being tied down to the autonomous case.

1. We examine Hamiltonians not containing linear terms, defined on one and the same $2n$-dimensional space, continuously differentiable with respect to the phase coordinates and depending continuously on time. The canonic transformations are treated as canonic automorphisms of this space. By the stability (or instability) of a Hamiltonian we mean the Liapunov stability (or Liapunov instability) of the zero solution of the corresponding Hamiltonian system for $t \geqslant 0$. The canonic transformation

$$k : q;\ p \to q'\ (t,\ q,\ p);\ p'\ (t,\ q,\ r);\ t \geqslant 0$$

is called a canonic Liapunov transformation if k leads every stable Hamiltonian into a stable one and an unstable one into an unstable one.

By L we denote the set of all canonic Liapunov transformations. Set L being a subgroup of the group of all canonic transformations, specifies an equivalence relation on the set of Hamiltonians being examined, namely, two Hamiltonians are equivalent (notation: $H_1 \sim H_2$) if we can find $k \in L$ and $k : H_1 \to H_2$. We denote the equivalence class of Hamiltonian H by $L\ (H)$. If H is autonomous, then $L_a\ (H)$ denotes the set of all autonomous Hamiltonians equivalent to H.

In connection with the investigation of stability in finite order it is necessary to modify somewhat the definitions presented. Firstly, all the Hamiltonians and canonic transformations are assumed in this case to be $(m + 1)$-times continuously differentiable with respect to the phase coordinates (m is the order to which the stability is to be investigated). Secondly, instead of k and H it is sufficient to examine the initial segments of their Taylor expansions up to m-th order. We denote them $k^{(m)}$ and $H^{(m)}$. Thirdly, the definition of equivalence is changed correspondingly: two Hamiltonians H_1 and H_2 are said to be m-equivalent ($H_1 \tilde{m} H_2$), if we can find a canonic transformation k : $k^{(m-1)} \in L$ leading H_1 into H_1' and $H_1'^{(m)} \equiv H_2^{(m)}$. We note that if $H_1 \tilde{m} H_2$ and $H_2^{(m)}$ is a stable Hamiltonian, then H_1 is said to be stable in the m-th order and the standard estimates [6] are valid for its phase flow. However, if $H_2^{(m)}$ is unstable, then in a majority of cases we can show that H_1 is unstable.

2. Let us consider certain criteria for k to belong to L. If the functions q', p', yielding a canonic transformation, are known in explicit form, we can apply the following.

A s s e r t i o n 1. Transformation $k \in L$ if and only if the following conditions are fulfilled: (a) $\| p'\ (t,\ q,\ p);\ q'\ (t,\ q,\ p) \|$ has an infinitely small upper bound at zero; (b) there exists a function $\delta\ (\varepsilon) > 0$, defined in some positive neighborhood of zero, such that $\|p'\ (t,\ q,\ p);\ q'\ (t,\ q,\ p) \| \geqslant \delta\ (\varepsilon)$ when $\|p;\ q\| \geqslant \varepsilon;\ t \geqslant 0$. Here $\| \cdot \|$ is the norm in phase space.

P r o o f. The sufficiency can be verified directly. We proove the necessity of condition (a). If it is not filfilled, then we can find sequences $q_i,\ p_i \to 0;\ t_i > 0$, and a constant $C > 0$ such that

$$\|\ p'\ (t_i,\ q_i,\ p_i);\ q'(t_i,\ q_i,\ p_i) \| > C \tag{2.1}$$

Let H_1 be the preimage of an identically zero Hamiltonian $H_2 \equiv 0$ under transformation k. Then q' and p' give the phase flow of H_1 for $t \geqslant 0$. In this case the instability of H_1 follows from (2.1); but since H_2 is stable, $k \bar{\in} L$. The necessity of (b) is proved analogously.

It turns out that condition (b) is a corollary of condition (a) for many important classes of canonic transformations. The three assertions presented below refer to this case.

Assertion 2. If k possesses the group property, i.e.

$$x(t_1+t_2,\ x)=x(t_1,\ x(t_2,\ x))\quad (x=(p,\ q)) \tag{2.2}$$

then $k\in L$ if and only if condition (a) of Assertion 1 is fulfilled.

Proof. From the one-to-oneness of k and from (2.2) it follows that $q'=q,\ p'=p$ for $t=0$ and, therefore, k yields a canonic dynamic system in the neighborhood of the phase space origin. Let us consider an arbitrary invariant for $t\geqslant 0$, open neighborhood R of the origin. R can be found because condition (a) is fulfilled and Poincaré theorem on the recurrence of points [7] is valid in R. We select constants $0<\delta_0<\varepsilon_0$ such that the sphere $S_{\varepsilon_0}=\{p,\ q\mid \|p,\ q\|\leqslant\varepsilon_0\}\in R$ and

$$p'(t,\ q,\ p);\ q'(t,\ q,\ p)\in S_{\varepsilon_0}\ \text{ for }\ t\geqslant 0;\ p,\ q\in S_{\delta_0} \tag{2.3}$$

We now assume that condition (b) is not fulfilled. Then we can find a point $q_0,\ p_0\in R$ such that, firstly, it is Poisson-recurrent for $t\geqslant 0$ [7] and, secondly, $\|q_0,\ p_0\|>\varepsilon_0$ and $\|p'(T,\ q_0,\ p_0);\ q'(T,\ q_0,\ p_0)\|\leqslant\delta_0$ for some $T>0$. Hence it follows that we can find $t_0>T$ for which

$$\|p'(t_0,\ q_0,\ p_0);\ q'(t_0,\ q_0,\ p_0)\|>\varepsilon_0 \tag{2.4}$$

If we denote $q_1=q'(T,\ q_0,\ p_0),\ p_1=p'(T,\ q_0,\ p_0)$, then from (2.2) and (2.4) we obtain

$$\|p'(t_0-T,\ q_1,\ p_1);\ \ q'(t_0-T,\ q_1,\ \ p_1)\|\geqslant\varepsilon_0;\ q_1,\ p_1\in S_{\delta_0}$$

whisch contradicts (2.3). The contradiction obtained proves what we required.

The next two assertions are presented without proofs, based on the specifying the canonic transformation by a generating function.

Assertion 3. In order that a linear canonic transformation be a Liapunov canonic transformation, it is necessary and sufficient that condition (a) of Assertion 1 be fulfilled.

Assertion 4. If a canonic transformation starts with an identity transformation, then $k^{(m)}\in L$ if and only if $u^{(m-1)}$ has an infinitely small upper bound at zero for $t\geqslant 0$, where u is the generating function for k.

3. Assertion 2 makes it possible to describe a sufficiently broad class of Liapunov canonic transformations. The phase flow of Hamiltonian H is denoted k_H.

Assertion 5. If H is a stable autonomous Hamiltonian, then $k_H\in L$.

Proof. By the definition of a stable Hamiltonian condition (a) of Assertion 1 is fulfilled for the canonic transformation k_H. Since H is autonomous, k_H satisfies the group property. Thus, all requirements of Assertion 2 are fulfilled and $k_H\in L$.

We present here several corollaries of the last assertion, which can be useful in the stability investigations of Hamiltonians. All the Hamiltonians below are assumed autonomous.

Corollary 1. If H is a stable Hamiltonian, then $H_1\equiv -H$ is stable as well.

Corollary 2. If $H(q,\ p)$ is a stable Hamiltonian, then $H_2\ (q,\ p)\equiv H\ (p,\ q)$ is stable as well.

Corollary 3. If H is a stable Hamiltonian, then the zero solution of the corresponding Hamiltonian system is Liapunov-stable also for $t\leqslant 0$.

Corollary 4. If $\{H,\ F\}=0$ ($\{,\}$ are the Poisson brackets) and F is a stable Hamiltonian, then $H\sim H+F$.

Assertion 5 makes possible a complete description of class $L_a(H)$ when H is stable.

Assertion 6. Let H be a stable autonomous Hamiltonian; then $L_a(H)$ consists of all stable autonomous Hamiltonians.

Proof. Let F be a stable Hamiltonian. The canonic transformation $k = k_F^{-1} \circ k_H$ takes H into F. Since H and F are stable and L is a group, $k \in L$ by Assertion 5. Hence $F \sim H$.

4. Consider the autonomous Hamiltonian

$$H = H_2 + H_3 + \ldots \tag{4.1}$$

In the stability investigation of such Hamiltonians it is convenient to use canonic transformations with a generating function of the form

$$u = qp' + \sum_i u_i(t, q, p') \tag{4.2}$$

where u_i is a homogeneous ith-order polynomial. By $L^m(H)$ we denote the set of all $F^{(m)}$ corresponding to those Hamiltonians F for which $H\tilde{m}F$ (see Sect. 1) in the class of canonic transformations with a generating function of form (4.2). The class $L_a^m(H)$ is determined from $L^m(H)$ in the same way that $L_a(H)$ was from $L(H)$.

The main question to be studied in Sect. 4 is the description of classes $L_a^m(H)$. The class $L_a^2(H)$ consists of only one Hamiltonian H_2. Therefore, we begin our analysis with $m=3$. Here, as usual, we assume that H_2 is a stable Hamiltonian; if this is not so, then, excepting the singular case of nonprime elementary divisors, H is unstable.

To study the class $L_a^3(H)$ it is sufficient to retain only u_3 in (4.2). As follows from Assertion 4, in order that $k^{(2)} \in L$, it is necessary and sufficient that

$$\sup_{t\geqslant 0} | u_3(t, q, p) | < +\infty \tag{4.3}$$

Let us consider a canonic transformation k with a generating function (4.2), where $u_i = 0$ for $i > 3$ and u_3 satisfies (4.3). Transformation k takes H into F defined by an identity in q, p', t

$$F\left(t, \frac{\partial u}{\partial p'}, p'\right) \equiv H\left(q, \frac{\partial u}{\partial q}\right) + \frac{\partial u}{\partial t}$$

Expanding both sides of the identity in a neighborhood of point q, p, we get that $F_2 \equiv H_2$, and F_3 is found from the relation

$$\partial u_3/\partial t + \{u_3, H_2\} = \Phi_3 \equiv F_3 - H_3 \tag{4.4}$$

Thus, for $F^{(3)} \equiv H_2 + F_3$ to belong to $L_a^3(H)$ it is necessary and sufficient that a cubic solution of Eq. (4.4) be found, satisfying (4.3). Here F_3 is independent of t. By direct computation we can verify that the general cubic solution of (4.4) is

$$u_3 = \int_0^t \Phi_3(k_{H_2}(-\tau, q, p))\, d\tau + v_3(t, q, p) \tag{4.5}$$

where v_3 is an arbitrary cubic first integral for H_2 ($\{v_3, H_2\} = 0$). Let us ascertain the conditions under which solution (4.5) of Eq. (4.4) will satisfy (4.3). The phase flow of the stable quadratic Hamiltonian H_2 is a linear conic transformation almost-periodic on the whole axis. Therefore, the first summand in expression (4.5) is bounded for $t \geqslant 0$ if and only if it is almost-periodic for $t \geqslant 0$ [8]. We denote it w_3. We note that the almost-periodicity of w_3 for $t \geqslant 0$ is equivalent to the almost-periodicity of w_3 for

$t \leqslant 0$. Indeed, the identity

$$w_3(t, k_{H_2}(t, q, p)) \equiv - w_3(-t, q, p)$$

is valid. The left-hand side of this identity, being a superposition of two functions almost-periodic for $t \geqslant 0$, is also almost-periodic for $t \geqslant 0$ [9]. The equality [8]

$$\bar{\Phi}_3 \equiv M\Phi_3 (k_{H_2}) = 0$$

serves as a necessary, and since w_3 is a trigonometrical polynomial, and also sufficient condition for the almost-periodicity of $w_3(-t, q, p)$. Here the overbar signifies averaging over the phase flow k_{H_2} for $t \geqslant 0$, while M is the operator of averaging over t for $t \geqslant 0$.

The second summand in (4.5) is represented as [10]

$$v_3(t, q, p) = \omega_3 (k_{H_2}(-t, q, p))$$

where ω_3 is an arbitrary third-order form. Therefore, w_3 is an almost-periodic function. Consequently, if $\bar{\Phi}_3 = 0$, then u_3 is an almost-periodic and, consequently, bounded function. Thus, the criterion for $F^{(3)}$ to belong to $L_a{}^3(H)$ can be written as

$$\bar{F}_3 = \bar{H}_3 \tag{4.6}$$

($\bar{H}_3$ is a cubic polynomial uniquely definable from H). Relation (4.6) signifies that there holds –

Assertion 7. Class $L_a{}^3(H)$ is an invariant set for the operator of averaging over the phase flow k_{H_2} and is completely defined by this condition.

To prove this we merely note that $\bar{H}_2 \equiv H_2$. The rest follows immediately from (4.6).

Assertion 8. If $F^{(3)} \in L_a{}^3(H)$ then we can find an autonomous canonic transformation establishing this membership.

This assertion has a rather unexpected corollary: the extension of the class of admissible canonic transformations up to the nonautonomous forms (4.1) does not extend the class of equivalent Hamiltonians obtainable.

Proof. Let $F^{(3)} \in L_a{}^3(H)$. In this case, according to (4.6), $\bar{F}_3 = \bar{H}_3$ and, as was ascertained above, the generating function u_3 defined by (4.5) is a cubic polynomial with almost-periodic coefficients. Therefore, $\partial u_3/\partial x_i$ $(x = (q, p))$ is a quadratic form with almost-periodic coefficients. Let us show that

$$M\frac{\partial u_3}{\partial x_i} \equiv \frac{\partial}{\partial x_i} M u_3 \tag{4.7}$$

To do this we write u_3 as

$$u_3 = \sum_{|l|=3} \alpha_l(t)\, x^l$$

$$l = (l_1, \ldots l_2), \quad x^l = x_1^{l_1} \ldots x_{2n}^{l_{2n}}, \quad |l| = \sum_{j=1}^{2n} l_j$$

Using the linearity of the averaging operator, we obtain

$$\frac{\partial}{\partial x_i} M u_3 = \frac{\partial}{\partial x_i} \sum_{|l|=3} (M\alpha_l)\, x^l = M\frac{\partial u_3}{\partial x_i}$$

We apply the averaging operator to identity (4.4), where the function u_3 is defined by

(4.5)

$$M\frac{\partial u_3}{\partial t}+M\{u_3, H_2\}\equiv M\Phi_3 \tag{4.8}$$

From (4.7) it follows that $M\{u_3, H_2\}=\{Mu_3, H_2\}$. In addition, because u_3 is bounded and Φ_3 is autonomous, $M\partial u_3/\partial t=0$ and $M\Phi_3=\Phi_3$. Therefore, (4.8) becomes the identity $\{Mu_3, H_2\}\equiv\Phi_3$. This identity signifies that Mu_3 is an autonomous solution of (4.4). Q.E.D.

Assertion 8 narrows the class of admissible canonic transformations of form (4.2) down to autonomous ones. It turns out that for obtaining the class $L_a^3(H)$ it is sufficient to examine not all the autonomous transformations but only those in which the generating function satisfies the condition $\bar{u}_3=0$.

Let us prove this. As we ascertained above, if $F^{(3)}\in L_a^3(H)$. then the canonic transformation with generating function $u=qp'+M_t^-w_3$, where w_3 is determined by (4.5), establishes the equivalence $H_3\sim F$. (The subscript of symbol M for the averaging operator denotes the variable with respect to which the averaging is carried out, while the superscript denotes the direction of the averaging: minus corresponds to $t\to-\infty$, plus corresponds to $t\to+\infty$.) We denote $u_3^*\equiv M_t^-w_3$. Using the strengthened theorem on the mean [8], we obtain

$$\bar{u}_3^*=M_\tau^+u_3^*(k_{H_2}(\tau, q, p))=M_\tau^+[M_t^-w_3(k_{H_2})]=M_\tau^+[M_t^-(w_3(t-\tau, q, p)-w_3(-\tau, q, p))]=M_\tau^+[u_3^*-w_3(-\tau, q, p)]=u_3^*-u_3^*=0$$

Thus, to obtain the whole class $L_a^3(H)$, it is sufficient to apply to H the set K_3 of autonomous canonic transformations of form (4.2) with $u_i=0$ for $i>3$ and $\bar{u}_3=0$. It can be shown that K_3 is the minimal set generating $L_a^3(H)$. Combining all the facts proved above, we can state

Assertion 9. To obtain the class $L_a^3(H)$ it is necessary and sufficient to apply the set K_3 of canonic transformations to H.

Set K_3 essentially depends upon the presence of third-order resonances in H_2. Thus, if H_2 does not have them at all, then it can be shown that K_3 contains all cubic forms, while at the other extreme case of $H_2\equiv 0$ it is easy to convince ourselves that K_3 contains only the identity transformation. Correspondingly, in the first case the class $L_a^3(H)$ includes all possible $F^{(3)}$, while in the second, only $F^{(3)}\equiv H^{(3)}$.

Let us pass on to the investigation of $L_a^4(H)$. Let $F^{(4)}\in L_a^4(H)$. Then, according to Assertion 4, $F^{(3)}\in L_a^3(H)$. For the determination of F_4 we obtain the equation

$$\frac{\partial u_4}{\partial t}+\{u_4, H_2\}=\Phi_4\equiv F_4-H_4-\Psi_4 \tag{4.9}$$

$$\Psi_4=-\frac{\partial F_3}{\partial q}\frac{\partial u_3}{\partial p}+\frac{\partial H_3}{\partial p}\frac{\partial u_3}{\partial q}-\frac{1}{2}\frac{\partial^2 H_2}{\partial q_1\partial q_2}\frac{\partial u_3}{\partial p_1}\frac{\partial u_3}{\partial p_2}$$

Equation (4.9) can be investigated in just the same way as Eq. (4.4). We state here the result of such an investigation.

Assertion 10. In order that $F^{(4)}\in L_a^4(H)$, it is necessary and sufficient that $\bar{F}^{(4)}=\bar{H}^{(4)}+\bar{\Psi}_4$.

Assertions of type 7 and 8 do not hold for $L_a^4(H)$ because, in general, Ψ_4 depends on t. But if we restrict ourselves in (4.2) to $u_3\in K_3$, then Assertion 7 carries over verbatim to $L_a^4(H)$, while in Assertion 8, instead of K_3 we should take the set K_4 of autonomous canonic transformations of form (4.2) with $u_i \equiv 0$ for $i > 4$ and $\bar{u}_3 = \bar{u}_4 = 0$. The investigation of classes $L_a{}^m (H)$ for $m > 4$ proceeds analogously. In the general case the condition for the membership: $F^m \in L_a{}^m (H)$ is the equality $\overline{F}^{(m)} = \overline{H}^{(m)} + \overline{\Psi}^{(m)}$, where $\Psi^{(m)}$ depends upon $F^{(i)}$, $H^{(i)}$ with $i < m$. Here, if we use only the canonic transformation from K_{m-1} for obtaining $F^{(m-1)}$, then for obtaining $F^{(m)}$ it is necessary and sufficient to apply to H the transformations from K_m, where K_m is the set of autonomous canonic transformations of form (4.2) with $u_i \equiv 0$ for $i > m$ and $\bar{u}_3 = \ldots = \bar{u}_m = 0$.

5. Let us explain how to find the normal form of an mth-order Hamiltonian H in the class $L_a{}^m (H)$. For this we present, first of all, a definition of normal form somewhat different from the usual one.

Definition. The Hamiltonian $h^{(m)} \equiv H_2 + h_3 + \ldots + h_m \in L_a{}^m (H)$, being in involution with H_2, is called the mth-order normal form of the Hamiltonian (4.1).

As follows, for example, from [1], this definition is equivalent to the standard one (defining the normal form as a polynomial with terms of a special structure) if H_2 has already been normalized, i.e. $H_2 = {}^1/_2 \Sigma \lambda_i (q_i^2 + p_i^2)$. In the general case the definition given is a generalization of the classical definition of normal form, which is not essential but is convenient for applications.

We start with the finding of the third-order normal form. From the definition given above and from (4.6) it follows that $h^{(3)} \equiv H_2 + h_3$; $\bar{h}_3 = \bar{H}_3$; h_3 is the first integral for H_2 and, therefore, $\bar{h}_3 = h_3$. Consequently, the third-order normal form is

$$h^{(3)} = H_2 + \bar{H}_3 \tag{5.1}$$

Formula (5.1) makes it possible to determine the third-order normal form right away, without having to find the normalizing transformation; from H_2 only stability is required. In particular, H_2 can have zero or equal natural frequencies. Note that the computation of $\bar{H}_3$ reduces to a simple computation of the integrals of sines and cosines. These cases were not examined in the usual approach to the normal form. An exception is the recent papers [11, 12] in which the normal form of a Hamiltonian system with two degrees of freedom is found in the presence of equal frequencies and certain applications of the normal form obtained to stability questions are studied.

For finding the fourth-order normal form, as in the case examined above, we obtain that

$$h^{(4)} = h^{(3)} + \overline{H}_4 + \overline{\Psi}_4 \tag{5.2}$$

where Ψ_4 is determined from (4.9). For finding h_4 in Ψ_4 we should set $F_3 \equiv \overline{H}_3$ and $u_3 \equiv w_3 \in K_3$. Analogously, we can show that for $m > 4$ the normal form is determined by the relation

$$h^{(m)} = h^{(m-1)} + \overline{H}_m + \overline{\Psi}_m \tag{5.3}$$

where Ψ_m is completely determined by $H^{(m-1)}$. We shall not derive here the general form of Ψ_m because of its awkwardness. We note merely that if we restrict ourselves to class K_m when finding the normalizing transformation, which is sufficient, as was ascertained above, then Ψ_m and, respectively, the normal form $h^{(m)}$ are determined uniquely. If, however, this condition is waived, then such uniqueness is not obtained, in general, because of the presence of an arbitrary first integral in formula (4.5).

The normalization method presented allows us to establish the close connection between the methods of normalization and of averaging. For this we apply the canonic transformation $k_{H_2}^{-1} \in L$ to the original Hamiltonian H. We then obtain $H \sim F$: $F_2 = 0$, $F_3 = H_3(k_{H_2})$, $F_4 = H_4(k_{H_2})$. Next, we make one more canonic transformation k_ε: $q, p \to \varepsilon^{-1}q, \varepsilon^{-1}p$ with valence ε^{-2}, where ε is a small parameter. We get that $F \sim G$: $G_2 = 0$, $G_3 = \varepsilon F_3$, $G_4 = \varepsilon^2 F_4$. The equation system with Hamiltonian G

$$q^{\cdot} = \frac{\partial G}{\partial p} \equiv \varepsilon \frac{\partial F_3}{\partial p} + \cdots, \quad p^{\cdot} = -\frac{\partial G}{\partial q} \equiv -\varepsilon \frac{\partial F_3}{\partial q}$$

is the standard system in the theory of averaging. Let us convince ourselves that we can apply Bogoliubov's fundamental averaging theorem [8] to it. To do this it is sufficient to show that $M_t{}^+\partial F_3/\partial x$ $(x = (q, p))$ exists uniformly in a neighborhood of zero. For which, in turn, it is sufficient that the family of functions

$$f_T(x) = \frac{1}{T}\int_0^T \frac{\partial F_3}{\partial x}\, dt$$

be uniformly bounded and equicontinuous. The first is obvious. While the second follows from the fact that $\partial^2 F_3 / \partial x_i \partial x_j$ are functions linear in x and almost-periodic in t and, therefore, are uniformly bounded.

Using the permutability of the operators of averaging and of differentiation (see Sect. 4), we can write the first-approximation averaged system as

$$q^{\cdot} = \varepsilon \frac{\partial}{\partial p} \overline{H}_3, \quad p = -\varepsilon \frac{\partial}{\partial q} \overline{H}_3$$

Carrying out the inverse canonic transformation $k_{H_2}^{-1} \circ k^{-1}$, we get that the Hamiltonian of the first-approximation equation system is equivalent (in the sense of Sect. 1) to the third-order normal form of the original Hamiltonian. An analogous analysis can be made for the higher orders.

6. Let us consider a mechanical example, being of independent interest, which illustrates the application of the results of Sect. 5 to the stability investigation of Hamiltonian systems. In [3], in the investigation of the stability of the steady-state rotations of a symmetrical artificial satellite in a circular orbit, the cases which in the parameter plane correspond to the boundaries of the stability domains were not considered. Let us analyze one such case here, namely, when $\alpha = \beta = 1$. This case corresponds to a relative equilibrium of the spherical artificial satellite. The Hamiltonian of this problem can be written as [3]

$$H = {}^1/_2\, p_1^2 + {}^1/_2\, p_2^2 + {}^1/_2\, q_2^2 + p_1 q_2 + {}^1/_8\, q_1^4 + {}^1/_2\, q_1^3 p_2 - {}^1/_{24}\, q_2^4 - {}^1/_6\, q_2^3 p_1 + {}^1/_2\, p_1^2 q_1^2 + {}^1/_4\, q_1^2 q_2^2 + {}^1/_2\, q_1 q_2^2 p_2$$

Using the method described in Sect. 5, we find the fourth-order normal form of this Hamiltonian. The natural frequencies of the linear systems equal zero and one. Therefore, the quadratic Hamiltonian is not normalizable in the usual sense. On the other hand, it is easy to write out its phase flow

$$\begin{aligned} q_1 &= -p_2^\circ \cos\nu + p_1^\circ + q_2^\circ \sin\nu + q_1^\circ + p_2^\circ \\ q_2 &= (q_2^\circ + p_1^\circ)\cos\nu + p_2^\circ \sin\nu - p_1^\circ \\ p_1 &= p_1^\circ, \ p_2 = p^\circ{}_2 \cos\nu - (p_1^\circ + q_2^\circ)\sin\nu \end{aligned} \tag{6.1}$$

Since $H_3 = 0$, then, according to (5.1), $h^{(3)} = H_2$ and $u_3 = 0$. Then from (5.2) we get that $h^{(4)} = H_2 + \overline{H}_4$. For finding $\overline{H}_4$ we should substitute (6.1) into H_4 and average over ν as $\nu \to +\infty$. After simple computations we find that

$$\overline{H}_4 = {}^1/_8\,(q_1^4 + p_1^4 + p_2^4 + 2\,q_1^2 p_1 + 2\,p_2^2 p_1 + 4\,q_1 p_1^2 p_2)$$

Finally, $h^{(4)}$ can be represented as

$$h^{(4)} = {}^1/_2\,[(p_1 + q_2)^2 + p_2^2] + {}^1/_8\,[(q_1 + p_2)^2 + p_1^2]^2 \tag{6.2}$$

From (6.2) we see that $h^{(4)}$ and, consequently, h are positive-definite functions. Hence the stability of H follows immediately, which signifies the Liapunov stability of the steady-state rotation being examined of the spherical artificial satellite on a circular orbit.

7. On the basis of the normalization method described we set up a table which allows us to write out the third-order normal form directly from the original Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} \alpha_i\,(q_i^2 + p_i^2) + \sum_{|k+l|=3} \alpha_{kl}\, q^k p^l \tag{7.1}$$

As was shown in Sect. 6, the third-order normal form of Hamiltonian (7.1) can be represented as

$$h^{(3)} = \frac{1}{2}\sum \alpha_i\,(q_i^2 + p_i^2) + \sum_{|k+l|=3} \alpha_{kl}\,\overline{q^k p^l}$$

The values of the averaged quantities are taken from the table in the column with the corresponding values of the integral n-vectors k and l (the table indicates the values of only the nonzero components of the integral vectors k and l)

$k=(k_1,\dots,k_n)$	$l=(l_1,\dots,l_n)$	$\overline{q^k p^l}$
$k_{i_1}=k_{i_2}=k_{i_3}=1$	$l=0$	$\langle ++++\rangle q_{i_1}q_{i_2}q_{i_3} + \langle -++-\rangle q_{i_1}p_{i_2}p_{i_3} + \langle -+-+\rangle p_{i_1}q_{i_2}p_{i_3} + \langle --++\rangle p_{i_1}p_{i_2}q_{i_3}$
$k_{i_1}=k_{i_2}=1$	$l_{i_3}=1$	$\langle ++++\rangle q_{i_1}q_{i_2}p_{i_3} + \langle +--+\rangle q_{i_1}p_{i_2}q_{i_3} + \langle +-+-\rangle p_{i_1}q_{i_2}q_{i_3} + \langle --++\rangle p_{i_1}p_{i_2}p_{i_3}$
$k_{i_1}=1$	$l_{i_2}=l_{i_3}=1$	$\langle ++++\rangle q_{i_1}p_{i_2}p_{i_3} + \langle -++-\rangle q_{i_1}q_{i_2}p_{i_3} + \langle ++--\rangle p_{i_1}q_{i_2}p_{i_3} + \langle +-+-\rangle p_{i_1}p_{i_2}q_{i_3}$
$k=0$	$l_{i_1}=l_{i_2}=l_{i_3}=1$	$\langle ++++\rangle p_{i_1}p_{i_2}p_{i_3} + \langle --++\rangle q_{i_1}q_{i_2}p_{i_3} + \langle -+-+\rangle q_{i_1}p_{i_2}q_{i_3} + \langle -++-\rangle p_{i_1}q_{i_2}q_{i_3}$

The symbol $\langle \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\rangle$ $(\varepsilon_i = \pm 1)$ denotes the algebraic sum $\varepsilon_1\chi_1 + \varepsilon_2\chi_2 + \varepsilon_3\chi_3 + \varepsilon_4\chi_4$ of the characteristic functions χ_i of the third-order resonances

$$\chi_i = \begin{cases} 0, & A_i \neq 0 \\ {}^1/_4, & A_i = 0 \end{cases}$$

$$A_1 = \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3}, \quad A_2 = \alpha_{i_1} + \alpha_{i_2} - \alpha_{i_3}$$

$$A_3 = \alpha_{i_1} - \alpha_{i_2} + \alpha_{i_3}, \quad A_4 = -\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3}$$

The author thanks V. V. Rumiantsev and the participants of the siminar directed by him for useful discussions of the results of this paper.

REFERENCES

1. Moser, J. K., Lectures on Hamiltonian Systems. Mem. Amer. Math. Soc., № 81, 1968.
2. Birkhoff, G. D., Dynamical Systems. New York, Amer. Math. Soc., 1927.
3. Markeev, A. P., Resonance effects and stability of steady-state rotations of an artificial satellite. Kosmicheskie Issledovaniia, Vol. 5, № 3, 1967.
4. Markeev, A. P., On the stability of the triangular libration points in the elliptic restricted three-body problem. PMM Vol. 34, № 2, 1970.
5. Khazin, L. G., On the stability of Hamiltonian systems in the presence of resonances. PMM Vol. 35, № 3, 1971.
6. Sigel, C. L., Vorlesungen über Himmelsmechanik. Berlin, Springer Verlag, 1957. (Russian translation: Lectures on Celestial Mechanics. Moscow, Izd. Inostr. Lit., 1959).
7. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations. Moscow-Leningrad, Gostekhizdat, 1949. (English translation: Qualitative Theory of Differential Equations. Princeton, Princeton Univ. Press, 1960).
8. Demidovich, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka", 1967.
9. Daletskii, Iu. L. and Krein, M. G., Stability of Solutions of Differential Equations in Banach Space. Moscow, "Nauka", 1970.
10. Stepanov, V. V., Course of Differential Equations. Moscow, Fizmatgiz, 1959.
11. Sokol'skii, A. G., On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. PMM Vol. 38, № 5, 1974.
12. Sokol'skii, A. G., On the stability of the Lagrange solutions of the restricted three-body problem with a critical mass ratio. PMM Vol. 39, № 2, 1975.

Translated by N. H. C.

UDC 531.31

SYNTHESIS OF DISCRETE VIBRATIONAL SYSTEMS WITH MAXIMALLY COMPRESSED SPECTRUM

PMM Vol. 39, № 4, 1975, pp. 614-620
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(Received October 1, 1973)

We propose a synthesis method for the parameter group of discrete vibrational systems, ensuring the maximal compression of the natural frequency spectrum. We give a method for solving two problems: (1) for a specified spectrum and definite part of the parameters find the values of the remaining parameters so that the lowest frequency would occupy the given position on the number axis and that the ratio of the highest frequency to the lowest would be minimal; (2) for a specified vibrational system obtain a system with maximally compressed spectrum at the expense of optimal vibration of a definite group of parameters.